

# GLOBAL SYMPLECTIC COORDINATES ON GRADIENT KÄHLER–RICCI SOLITONS

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**ABSTRACT.** A classical result of D. McDuff [14] asserts that a simply-connected complete Kähler manifold  $(M, g, \omega)$  with non positive sectional curvature admits global symplectic coordinates through a symplectomorphism  $\Psi: M \rightarrow \mathbb{R}^{2n}$  (where  $n$  is the complex dimension of  $M$ ), satisfying the following property (proved by E. Ciriza in [4]): the image  $\Psi(T)$  of any complex totally geodesic submanifold  $T \subset M$  through the point  $p$  such that  $\Psi(p) = 0$ , is a complex linear subspace of  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ . The aim of this paper is to exhibit, for all positive integers  $n$ , examples of  $n$ -dimensional complete Kähler manifolds with non-negative sectional curvature globally symplectomorphic to  $\mathbb{R}^{2n}$  through a symplectomorphism satisfying Ciriza’s property.

## 1. INTRODUCTION

D. McDuff [14] (see also [1]) proved a global version of Darboux theorem for  $n$ -dimensional complete and simply-connected Kähler manifolds with nonpositive sectional curvature. She shows that there exists a diffeomorphism  $\Psi: M \rightarrow \mathbb{R}^{2n} = \mathbb{C}^n$  satisfying  $\Psi^*(\omega_0) = \omega$ , where  $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$  is the standard symplectic form on  $\mathbb{R}^{2n}$ . The interest for these kind of questions comes, for example, after Gromov’s discovery [9] of the existence of exotic symplectic structures on  $\mathbb{R}^{2n}$ . E. Ciriza [4] (see also [3] and [5]) proves that the image  $\Psi(T)$  of any complete complex and totally geodesic submanifold  $T$  of  $M$  passing through the point  $p$  such that  $\Psi(p) = 0$ , is a complex linear subspace of  $\mathbb{C}^n$ . A global symplectomorphism satisfying this property has been constructed by the first author and A. J. Di Scala [7] (see also [8]) for Hermitian symmetric spaces of noncompact type and by the authors of the present paper for the Calabi’s inhomogeneous Kähler–Einstein metric on tubular domains (cfr. [12]). It is then natural and interesting to investigate the existence of positively curved complete Kähler manifolds globally symplectomorphic to  $\mathbb{R}^{2n}$  through a symplectomorphic satisfying the above Ciriza’s property. In this paper we construct explicit

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global symplectic coordinates for the positively curved complete gradient Kähler–Ricci solitons built by H. D. Cao in [2]. Moreover, we exhibit, for all positive integres  $n$ , an example of gradient Kähler–Ricci solitons (the product of  $n$  copies of the Cigar soliton) where Ciriza’s property holds true. Our results are summarized in the following two theorems (see next section for details and terminology).

**Theorem 1.** *A gradient Kähler–Ricci soliton  $(\mathbb{C}^n, \omega_{RS})$  is globally symplectomorphic to  $(\mathbb{R}^{2n}, \omega_0)$ .*

**Theorem 2.** *Let  $(\mathbb{C}^n, \omega_{C,n})$  be the the product of  $n$  copies of the Cigar soliton. Then there exists a symplectomorphism  $\Psi_{C,n} : (\mathbb{C}^n, \omega_{C,n}) \rightarrow (\mathbb{R}^{2n}, \omega_0)$ , with  $\Psi_{C,n}(0) = 0$ , taking complete complex totally geodesic submanifolds through the origin to complex linear subspaces of  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ .*

The paper consists of two other sections containing respectively the basic material on gradient Kähler–Ricci solitons and the proofs of the main results.

## 2. GRADIENT KÄHLER–RICCI SOLITONS

We recall here what we need about the gradient Kähler–Ricci solitons described by H-D. Cao in [2] (to whom we refer for references and further details). Let  $g_{RS}$  be the Kähler metric on  $\mathbb{C}^n$  generated by the radial Kähler potential  $\Phi(z, \bar{z}) = u(t)$ , where for all  $t \in (-\infty, +\infty)$ ,  $u$  is a smooth function of  $t = \log(\|z\|^2)$  and as  $t \rightarrow -\infty$  it has an expansion:

$$u(t) = a_0 + a_1 e^t + a_2 e^{2t} + \dots, \quad a_1 = 1. \quad (1)$$

Denote by  $\omega_{RS} = \frac{i}{2} \partial \bar{\partial} \Phi$  the Kähler form associated to  $g_{RS}$ . If  $u$  satisfies the equation:

$$(u')^{n-1} u'' e^{u'} = e^{nt},$$

then the conditions:

$$u'(t) > 0, \quad u''(t) > 0, \quad \forall t \in (-\infty, +\infty), \quad (2)$$

$$\lim_{t \rightarrow +\infty} \frac{u'(t)}{t} = n, \quad \lim_{t \rightarrow +\infty} u''(t) = n. \quad (3)$$

are fulfilled and  $(\mathbb{C}^n, \omega_{RS})$  is a gradient Kähler–Ricci soliton. The metric  $g_{RS}$  is complete and positively curved and for  $n = 1$  one recovers the Cigar metric on  $\mathbb{C}$  whose associated Kähler form reads:

$$\omega_C = \frac{dz \wedge d\bar{z}}{1 + |z|^2},$$

which was introduced by Hamilton in [10] as first example of Kähler–Ricci soliton on non-compact manifolds. Observe that a Kähler potential for  $\omega_C$  is given by (see also [15]):

$$\Phi_C = \int_0^{|z|} \frac{\log(1 + s^2)}{s} ds.$$

Furthermore, in this case the Riemannian curvature reads:

$$R = \frac{1}{(1 + |z|^2)^3}. \quad (4)$$

It is interesting observing that the Kähler metric  $\omega_{C,n}$  on  $\mathbb{C}^n = \frac{i}{2}\partial\bar{\partial}\Phi_{C,n}$  defined as product of  $n$  copies of Cigar metric  $\omega_C$ , satisfies  $\Phi_{C,n} = \Phi_C \oplus \dots \oplus \Phi_C$  and it is still a complete and positively curved (i.e. with non-negative sectional curvature) gradient Kähler–Ricci soliton, namely it satisfies (1), (2) and (3) above. In particular its Riemannian tensor satisfies  $R_{i\bar{j}k\bar{l}} = 0$  whenever one of the indexes is different from the others and by (4) it is easy to see that the nonvanishing components are given by:

$$R_{j\bar{j}j\bar{j}} = \frac{1}{(1 + |z_j|^2)^3}. \quad (5)$$

### 3. PROOF OF THE MAIN RESULTS

In [13] the first author of the present paper, jointly with F. Zuddas, proved the following result on the existence of a symplectomorphism between a rotation invariant Kähler manifold of complex dimension  $n$  and  $(\mathbb{R}^{2n}, \omega_0)$ . For the readers convenience, we summarize here that result and the proof in the case when the manifold is  $\mathbb{C}^n$ . This will be the main ingredient in the proof of our main results.

**Lemma 3.** *Let  $\omega_\Phi = \frac{i}{2}\partial\bar{\partial}\Phi$  be a rotation invariant Kähler form on  $\mathbb{C}^n$  i.e. the Kähler potential only depends on  $|z_j|^2$ ,  $j = 1, \dots, n$ .<sup>1</sup> If*

$$\frac{\partial\Phi}{\partial|z_k|^2} \geq 0, \quad k = 1, \dots, n. \quad (6)$$

*then the map:*

$$\Psi : (M, \omega_\Phi) \rightarrow (\mathbb{C}^n, \omega_0), \quad z = (z_1, \dots, z_n) \mapsto (\psi_1(z)z_1, \dots, \psi_n(z)z_n),$$

*where*

$$\psi_j = \sqrt{\frac{\partial\Phi}{\partial|z_j|^2}}, \quad j = 1, \dots, n,$$

*is a symplectic immersion. If in addition:*

$$\lim_{z \rightarrow +\infty} \sum_{j=1}^n \frac{\partial\Phi}{\partial|z_j|^2} |z_j|^2 = +\infty, \quad (7)$$

*then  $\Psi$  is a global symplectomorphism.*

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<sup>1</sup>Notice that the rotation invariant condition on the potential  $\Phi$  is more general then the radial one which requires  $\Phi$  depending only on  $|z_1|^2 + \dots + |z_n|^2$ .

*Proof.* Assume condition (6) holds true. Let us prove first that  $F^*\omega_0 = \omega$ . We have:

$$\begin{aligned}\Psi^*\omega_0 &= \frac{i}{2} \sum_{j=1}^n d\Psi_j \wedge d\bar{\Psi}_j \\ &= \sum_{j=1}^n \left( \frac{\partial\Psi_j}{\partial z_j} dz_j + \frac{\partial\Psi_j}{\partial \bar{z}_j} d\bar{z}_j \right) \wedge \left( \frac{\partial\bar{\Psi}_j}{\partial z_j} dz_j + \frac{\partial\bar{\Psi}_j}{\partial \bar{z}_j} d\bar{z}_j \right) \\ &= \sum_{j,k=1}^n \left( \left| \frac{\partial\Psi_j}{\partial z_j} \right|^2 - \left| \frac{\partial\bar{\Psi}_j}{\partial z_j} \right|^2 \right) dz_j \wedge d\bar{z}_j\end{aligned}$$

Since

$$\frac{\partial\Psi_j}{\partial z_j} = \frac{\partial\psi_j}{\partial z_j} z_j + \psi_j, \quad \frac{\partial\Psi_j}{\partial \bar{z}_j} = \frac{\partial\psi_j}{\partial \bar{z}_j} z_j,$$

and

$$\frac{\partial\psi_j}{\partial z_j} = \frac{1}{2} \psi_j^{-1} \left( \frac{\partial^2 \Phi}{\partial |z_j|^4} \right) \bar{z}_j,$$

it follows:

$$\begin{aligned}\Psi^*\omega_0 &= \sum_{j=1}^n \left( \left| \frac{\partial\psi_j}{\partial z_j} z_j + \psi_j \right|^2 - \left| \frac{\partial\psi_j}{\partial z_j} \right|^2 |z_j|^2 \right) dz_j \wedge d\bar{z}_j \\ &= \sum_{j=1}^n \left( \frac{\partial\psi_j}{\partial z_j} \psi_j z_j + \frac{\partial\psi_j}{\partial \bar{z}_j} \psi_j \bar{z}_j + \psi_j^2 \right) dz_j \wedge d\bar{z}_j \\ &= \sum_{j=1}^n \left( \left( \frac{\partial^2 \Phi}{\partial |z_j|^4} \right) |z_j|^2 + \left( \frac{\partial \Phi}{\partial |z_j|^2} \right) \right) dz_j \wedge d\bar{z}_j \\ &= \sum_{j=1}^n \frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_j} dz_j \wedge d\bar{z}_j.\end{aligned}$$

Observe now that since  $\omega$  and  $\omega_0$  are non-degenerate, it follows by the inverse function theorem that  $\Psi$  is a local diffeomorphism. If in addition condition (7) holds true, then  $\Psi$  is a proper map and hence a global diffeomorphism.  $\square$

We are now in the position of proving Theorem 1.

*Proof of Theorem 1.* Let  $\Phi(z, \bar{z}) = u(t)$ , where  $u(t)$  is given by (1). Then for all  $j = 1, \dots, n$

$$\frac{\partial \Phi}{\partial |z_j|^2} = \frac{\partial \Phi}{\partial ||z||^2} = \frac{u'(\log(||z||^2))}{||z||^2},$$

which is greater than zero for all  $||z||^2 \neq 0$  by (2), and evaluated at  $||z||^2 = 0$  gives the value 1 by (1). Notice now that by the first of the limit conditions

given in (3) it follows that condition (7) in Lemma 3 holds true. Therefore by Lemma 3 the map:

$$F: (\mathbb{C}^n, g_{RS}) \rightarrow (\mathbb{R}^{2n}, g_0), \quad z = (z_1, \dots, z_n) \mapsto \sqrt{\frac{u'(\log(\|z\|^2))}{\|z\|^2}} (z_1, \dots, z_n),$$

is the desired global symplectomorphism.  $\square$

In order to prove Theorem 2 we need the following lemma which classifies all totally geodesic submanifolds of  $(\mathbb{C}^n, \omega_{C,n})$  through the origin.

**Lemma 4.** *Let  $S$  be a totally geodesic complex submanifold (of complex dimension  $k$ ) of  $(\mathbb{C}^n, \omega_{C,n})$ . Then, up to unitary transformation of  $\mathbb{C}^n$ ,  $S = (\mathbb{C}^k, \omega_{C,k})$ .*

*Proof.* Let us first prove the statement for  $n = 2$ . For  $k = 0, 2$  there is nothing to prove, thus fix  $k = 1$ . Let

$$f: (S, \tilde{\omega}) \hookrightarrow (\mathbb{C}^2, \omega_{C,2}), \quad f(z) = (f_1(z), f_2(z)).$$

be a totally geodesic embedding of a 1-dimensional complex manifold  $(S, \tilde{\omega})$  into  $(\mathbb{C}^2, \omega_{C,2})$ . By  $\tilde{\omega} = f^*(\omega_{C,2})$  we get:

$$\tilde{\omega} = \frac{i}{2} \left( \left| \frac{\partial f_1}{\partial z} \right|^2 \frac{1}{1 + |f_1(z)|^2} + \left| \frac{\partial f_2}{\partial z} \right|^2 \frac{1}{1 + |f_2(z)|^2} \right) dz \wedge d\bar{z}. \quad (8)$$

Let  $\tilde{R}$ ,  $R_C$  be the curvature tensor of  $(S, \tilde{\omega})$  and  $(\mathbb{C}^2, \omega_C)$  respectively. Since  $(S, \tilde{\omega})$  is totally geodesic in  $(\mathbb{C}^2, \omega_C)$  we have

$$\tilde{R}(X, JX, X, JX) = R_C(X, JX, X, JX)$$

for all the vector fields  $X$  on  $S$  (see e.g. [11, p. 176]). Taking  $X = \partial/\partial z$ , we have:

$$\tilde{R} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) = -\frac{\partial^2 \tilde{g}}{\partial z \partial \bar{z}} + \tilde{g}^{-1}(z) \left| \frac{\partial \tilde{g}(z)}{\partial z} \right|^2,$$

where  $\tilde{g}$  is the Kähler metric associated to  $\tilde{\omega}$ , i.e.

$$\tilde{g} = \left| \frac{\partial f_1}{\partial z} \right|^2 \frac{1}{1 + |f_1(z)|^2} + \left| \frac{\partial f_2}{\partial z} \right|^2 \frac{1}{1 + |f_2(z)|^2}.$$

Further, since the vector field  $\frac{\partial}{\partial z}$  corresponds through  $df$  to  $\frac{\partial f_1}{\partial z} \frac{\partial}{\partial z_1} + \frac{\partial f_2}{\partial z} \frac{\partial}{\partial z_2}$ , by (5) we get:

$$R_C \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) = \left| \frac{\partial f_1}{\partial z} \right|^4 \frac{1}{(1 + |f_1(z)|^2)^3} + \left| \frac{\partial f_2}{\partial z} \right|^4 \frac{1}{(1 + |f_2(z)|^2)^3}.$$

Since

$$\frac{\partial \tilde{g}}{\partial z} = \sum_{j=1}^2 \left( \frac{2}{1 + |f_j(z)|^2} \frac{\overline{\partial f_j}}{\partial z} \frac{\partial^2 f_j}{\partial z^2} - \left| \frac{\partial f_j}{\partial z} \right|^2 \frac{\bar{f}_j}{(1 + |f_j|^2)^2} \frac{\partial f_j}{\partial z} \right),$$

$$\begin{aligned} \frac{\partial^2 \tilde{g}}{\partial z \partial \bar{z}} = & \sum_{j=1}^2 \left[ \left| \frac{\partial f_j}{\partial z} \right|^4 \frac{2|f_j|^2}{(1+|f_j(z)|^2)^3} + \left| \frac{\partial^2 f_j}{\partial z^2} \right|^2 \frac{1}{1+|f_j(z)|^2} + \right. \\ & \left. - \frac{1}{(1+|f_j|^2)^2} \left( \bar{f}_j \left( \frac{\partial f_j}{\partial z} \right)^2 \frac{\partial^2 \bar{f}_j}{\partial z^2} + \left| \frac{\partial f_j}{\partial z} \right|^4 + f_j \left( \frac{\partial \bar{f}_j}{\partial z} \right)^2 \frac{\partial^2 f_j}{\partial z^2} \right) \right] \end{aligned}$$

after a long but straightforward computation, we get that  $\tilde{R} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) - R_C \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)$  assumes the form:

$$\frac{-|A(f_1, f_2)|^2}{\left( \left| \frac{\partial f_1}{\partial z} \right|^2 (1+|f_2|^2) + \left| \frac{\partial f_2}{\partial z} \right|^2 (1+|f_1|^2) \right) (1+|f_1|^2)^2 (1+|f_2|^2)^2},$$

where

$$\begin{aligned} A(f_1, f_2) = & \left( \frac{\partial^2 f_2}{\partial z^2} \frac{\partial f_1}{\partial z} - \frac{\partial^2 f_1}{\partial z^2} \frac{\partial f_2}{\partial z} \right) (1+|f_1|^2)(1+|f_2|^2) + \\ & + \left( \frac{\partial f_1}{\partial z} \right)^2 \frac{\partial f_2}{\partial z} \bar{f}_1 (1+|f_2|^2) - \left( \frac{\partial f_2}{\partial z} \right)^2 \frac{\partial f_1}{\partial z} \bar{f}_2 (1+|f_1|^2). \end{aligned}$$

Thus,  $\tilde{R} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) - R_C \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) = 0$  iff  $A(f_1, f_2) = 0$ , i.e. iff

$$\begin{aligned} & \frac{\partial f_1}{\partial z} (1+|f_2|^2) \left( \frac{\partial^2 f_2}{\partial z^2} (1+|f_1|^2) + \frac{\partial f_1}{\partial z} \frac{\partial f_2}{\partial z} \bar{f}_1 \right) = \\ & = \frac{\partial f_2}{\partial z} (1+|f_1|^2) \left( \frac{\partial^2 f_1}{\partial z^2} (1+|f_2|^2) + \frac{\partial f_2}{\partial z} \frac{\partial f_1}{\partial z} \bar{f}_2 \right), \end{aligned} \tag{9}$$

which is verified whenever one between  $f_1(z)$  and  $f_2(z)$  is constant (and thus zero since we assume  $f(0, 0) = 0$ ), or when  $f_1(z) = f_2(z)$ . In order to prove that these are the only solutions, write (9) as

$$\frac{\partial f_1}{\partial z} (1+|f_2|^2) \frac{\partial}{\partial z} \left( \frac{\partial f_2}{\partial z} (1+|f_1|^2) \right) = \frac{\partial f_2}{\partial z} (1+|f_1|^2) \frac{\partial}{\partial z} \left( \frac{\partial f_1}{\partial z} (1+|f_2|^2) \right).$$

Assuming  $f_1, f_2$  not constant, it leads to the equation:

$$\left( \frac{\frac{\partial f_1}{\partial z} (1+|f_2|^2)}{\frac{\partial f_2}{\partial z} (1+|f_1|^2)} \right)' \left( \frac{\partial f_2}{\partial z} (1+|f_1|^2) \right)^2 = 0,$$

which implies that for some complex constant  $\lambda \neq 0$ ,

$$\frac{\partial f_1}{\partial z} (1+|f_2|^2) = \lambda \frac{\partial f_2}{\partial z} (1+|f_1|^2), \tag{10}$$

that is:

$$\frac{\partial \log f_1}{\partial z} \bar{f}_1 = \lambda \frac{\partial \log f_2}{\partial z} \bar{f}_2.$$

Comparing the antiholomorphic parts we get  $\bar{f}_1 = \alpha \bar{f}_2$ , for some complex constant  $\alpha$ . Substituting in (10) we get:

$$\alpha(1+|f_2|^2) = \lambda(1+|\alpha|^2|f_2|^2).$$

Since  $f(0, 0) = 0$ , from this last equality follows  $\alpha = \lambda$  and thus immediately  $|\alpha|^2 = 1$ . We have been proven that a totally geodesic submanifold of  $(\mathbb{C}^2, \omega_{C,2})$  is, up to unitary transformation of  $\mathbb{C}^2$ ,  $(\mathbb{C}, \omega_C)$  realized either via the map  $z \mapsto (f_1, 0)$  (or equivalently  $z \mapsto (0, f_1)$ ) or via  $z \mapsto (f_1(z), \alpha f_1(z))$ , with  $|\alpha|^2 = 1$ .

Assume now  $S$  to be a  $k$ -dimensional complete totally geodesic complex submanifold of  $(\mathbb{C}^n, \omega_{C,n})$  and let  $\pi_j$ ,  $j = 1, \dots, n$ , be the projection into the  $j$ th  $\mathbb{C}$ -factor in  $\mathbb{C}^n$ ,  $\pi_{jk}$ ,  $j, k = 1, \dots, n$ , the projection into the space  $\mathbb{C}^2$  corresponding to the  $j$ th and  $k$ th  $\mathbb{C}$ -factors. Since  $\pi_j(S)$ ,  $j = 1, \dots, n$ , is totally geodesic into  $(\mathbb{C}, \omega_C)$ , it is either a point or the whole  $\mathbb{C}$ . Thus, up to unitary transformation of the ambient space, we can assume  $S$  to be of the form:

$$(z_1, \dots, z_k) \mapsto (0, \dots, 0, h_{11}(z_1), \dots, h_{1r}(z_1), \dots, h_{k1}(z_k), \dots, h_{ks}(z_k)). \quad (11)$$

Since also the projections  $\pi_{jk}(S)$  have to be totally geodesic into  $(\mathbb{C}^2, \omega_{C,2})$ , by what we have proven for  $n = 2$ , we can reduce (11) into the form:

$$(z_1, \dots, z_k) \mapsto (0, \dots, 0, h_1(z_1), \dots, \alpha_r h_1(z_1), \dots, h_k(z_k), \dots, \alpha_s h_k(z_k)),$$

where  $|\alpha_t|^2 = 1$  for all  $t$  appearing above. Thus, either  $S = (\mathbb{C}^k, \omega_{C,k})$  or  $S$  is a  $k$  dimensional diagonal, which with a suitable unitary transformation can be written again as  $(\mathbb{C}^k, \omega_{C,k})$ , and we are done.  $\square$

*Proof of Theorem 2.* The existence of a global symplectomorphism  $\Psi_{C,n} : (\mathbb{C}^n, \omega_{C,n}) \rightarrow (\mathbb{R}^{2n}, \omega_0)$  is guaranteed again by Lemma 3. In fact for all  $j = 1, \dots, n$

$$\begin{aligned} \frac{\partial}{\partial |z_j|^2} \Phi_{C,n} &= 2 \frac{\partial}{\partial |z_j|^2} \sum_{j=1}^n \int_0^{|z_j|} \frac{\log(1+s^2)}{s} ds \\ &= \frac{1}{|z_j|} \frac{d}{d|z_j|} \int_0^{|z_j|} \frac{\log(1+s^2)}{s} ds = \frac{\log(1+|z_j|^2)}{|z_j|^2} > 0. \end{aligned}$$

Moreover, condition (7) in Lemma 3 is fulfilled by:

$$\lim_{z \rightarrow +\infty} |z_j|^2 \sum_{j=1}^n \frac{\partial \Phi_{C,n}}{\partial |z_j|^2} = \lim_{z \rightarrow +\infty} \sum_{j=1}^n \log(1+|z_j|^2) = +\infty.$$

Thus by Lemma 3 the map:

$$\Psi_{C,n} : (\mathbb{C}^n, \omega_{C,n}) \rightarrow (\mathbb{R}^{2n}, \omega_0), \quad z = (z_1, \dots, z_n) \mapsto (\psi_1(z_1)z_1, \dots, \psi_n(z_n)z_n),$$

with

$$\psi_j = \sqrt{\frac{\log(1+|z_j|^2)}{|z_j|^2}},$$

is a global symplectomorphism.

In order to prove the second part of the theorem, let  $S$  be a  $k$  dimensional totally geodesic complex submanifold of  $(\mathbb{C}^n, \omega_{C,n})$  through the origin, which by Lemma 4 is given by  $(\mathbb{C}^k, \omega_{C,k})$ . The image  $\Psi_{C,n}(S)$  is of the form:

$$\left( \sqrt{\frac{\log(1 + |z_1|^2)}{|z_1|^2}} z_1, \dots, \sqrt{\frac{\log(1 + |z_k|^2)}{|z_k|^2}} z_k, 0, \dots, 0 \right) \simeq \mathbb{C}^k,$$

concluding the proof.  $\square$

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